L03 Jan8 Open Closed

Friday, January 2, 2015 4:55 PM

A topological space
$(X, \mathcal{T})$ closed under union & finite intersection

A set $G \subset X$ is open if $G \in \mathcal{T}$

Qu. How to define interior point?

Hint. Think about nbhd.

$x \in A$ is an interior point of A if
$\exists\, U \in \mathcal{T}$ such that $x \in U \subset A$

Notation. $x \in \mathring{A}$ or $x \in \mathrm{Int}(A)$

Theorem $\mathring{A}$ is the largest open subset of A

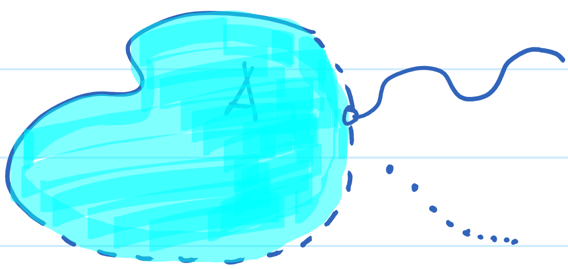
Essentially, prove $\mathring{A} = \bigcup \{ G \subset A : G \in \mathcal{T} \}$
Also get $\mathring{A} \in \mathcal{T}$ for all $A \subset X$.

Theorem. G is open
$\iff$ Every $x \in G$ is an interior point
$\iff G = \mathring{G} \iff G \subset \mathring{G}$

Qu. What is $(\mathring{A})^\circ = \mathrm{Int}(\mathrm{Int}(A))$?

The answer is given above

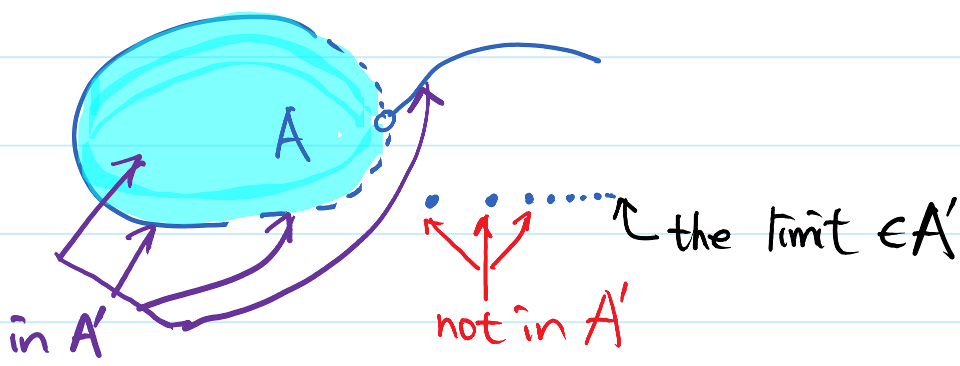
For such a picture of
$A \subset \mathbb{R}^2$,



besides interior points, there are other properties. What can you think of?

Definition Let $A \subset X$, $x \in X$ is a cluster point of A or accumulation point or limit point if for all $U \in \mathcal{N}_x$, nbhd of x, $U \cap A \setminus \{x\} \neq \emptyset$

Notation $x \in A'$, derived set of A



Definition The closure of A is the set

$$\bar{A} \text{ or } Cl(A) = A \cup A'$$

Fact $x \in \bar{A} \iff \forall$ nbhd U of x, $U \cap A \neq \emptyset$

Definition $x \in X$ is a frontier or boundary point of A if for each nbhd U of x,

$U \cap A \neq \emptyset$ and $U \cap (X \setminus A) \neq \emptyset$

Obviously $Frt(A) = \bar{A} \cap \overline{(X \setminus A)}$

Definition $A \subset X$ is closed if $X \setminus A \in \mathcal{T}$

A trivial consequence from T1 and T2

(T1) Any intersection of closed sets is closed

(T2) A finite union of closed sets is closed

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Theorem $F \subset X$ is closed

$\Leftrightarrow F = \overline{F} \Leftrightarrow F \supset \overline{F} \Leftrightarrow F \supset F'$

Hint. Write $x \notin \overline{A}$ in terms of nbhds.

$x \notin \overline{A} \Leftrightarrow \exists$ nbhd U of x, $U \cap A = \emptyset$

i.e. $U \subset X \setminus A$

Thus, $X \setminus \overline{A} = (X \setminus A)^\circ$

Equivalently, $\overline{A} = X \setminus (X \setminus A)^\circ$

This fact leads to the above theorem and below.

Theorem $\overline{A}$ is the smallest closed set $\supset A$